Appendices

A Laplacian Characterization

A.1 Generic term of the Laplacian matrix

Let Δ be the Laplacian of an hypergraph $\mathcal{H} = (N, E)$.

We write $\Delta = G^T G$ where G is the matrix of the unnormalized gradient for some ordering of the nodes $v_1, \ldots, v_{|N|}$ and some ordering of the hyperedges $h_1, \ldots, h_{|E|}$. We have:

$$G_{i,j} = \epsilon_{h_i}(v_j)\sqrt{w_{h_i}(v_j)}$$

where:

$$\epsilon_h(v) = \begin{cases} 1 \text{ if } v \in t_h \\ -1 \text{ if } v \in s_h \\ 0 \text{ otherwise} \end{cases}$$

We get directly the generic term of the Laplacian matrix:

$$\Delta_{i,j} = \sum_k G_{k,i} G_{k,j} = \sum_k P_{h_k}(v_i, v_j)\sqrt{w_{h_k}(v_i)}\sqrt{w_{h_k}(v_j)}$$

A.2 Proof of theorem 1

We are going to proove here the theorem 1 which states that a matrix M is the Laplacian of an hypergraph if and only if it is symmetric semi-definite positive and $\mathbf{1} \in Ker(M)$.

Proof.
$\boxed{\Leftarrow}$
If $M \in \mathcal{M}_n(\mathbb{R})$ is symmetric semi-definite positive, we can write $M = G^T G$ where G is a square root of M. We have $\mathbf{1} \in Ker(M)$ so:

$$\begin{aligned} \mathbf{1}^T M \mathbf{1} &= 0 \\ (G\mathbf{1})^T (G\mathbf{1}) &= 0 \\ \|G\mathbf{1}\|^2 &= 0 \end{aligned}$$

and $\mathbf{1} \in Ker(G)$.
We define an hypergraph $\mathcal{H}$ on n edges $N = \{v_1, \ldots, v_n\}$ using the following rule: for the i-th line of G, we create the hyperedge h_i where for all $v_j \in N$, $w_{h_i}(v_j) = |G_{i,j}|^2$. The nodes with positive weights will be assigned to one end and those with negative weights to the other end.

The weight equilibrium property holds for all these hyperedges because the lines of G have a null sum ($\mathbf{1} \in Ker(G)$). We can easily verify that G is a gradient matrix for $\mathcal{H}$. The transposition of this matrix gives a matrix of $-div$ (hermitian adjoint) so we can write the unnormalized Laplacian:

$$\Delta = \mathcal{M}(grad)^T \mathcal{M}(grad) = M$$

,

which concludes the proof.

$\boxed{\Rightarrow}$
Let Δ be the Laplacian of an hypergraph $H = (N, E)$. Δ is symmetric and the sum of its i-th line can be written as:

$$\begin{aligned} S_i &= \sum_j \sum_k \epsilon_{h_k}(v_i)\epsilon_{h_k}(v_j)\sqrt{w_{h_k}(v_i)}\sqrt{w_{h_k}(v_j)} \\ &= \sum_k \epsilon_{h_k}(v_i)\sqrt{w_{h_k}(v_i)} \quad \underbrace{\sum_j \epsilon_{h_k}(v_j)\sqrt{w_{h_k}(v_j)}}_{=0 \quad \text{(weight equilibrium in } h_k)} \\ &= 0 \end{aligned}$$

So Δ is $\textsc{sms}_0$. Let's now consider the gradient matrix G. As above, we have $\Delta = G^T G$. The singular value decomposition of this matrix gives:

$$G = U \Sigma V^T$$

$\Delta = V \Sigma^2 V^T$ has a positive spectrum and is semi-definite positive, which concludes the proof.

B Interpretation of Laplacian Distances

We are going to give in this section some important proofs that allow us to interpret the resistance distances linked to the Laplacian matrices. The following results are valid as soon as $Ker(\Delta) = Span(\mathbf{1})$ (rank$(\Delta) = |N| - 1$), i.e. when the related hypergraph is *connected*.

Consider the hypergraph as an equivalent electrical network. The Poisson's law for the electric potential V in the static case is $\Delta V = \frac{-\rho}{\epsilon_0}$ where ρ is the charge density. This law can be used in our discrete analysis framework if we consider V as a real-valued node function. Let us define the input function of our network as the real-valued function $\boldsymbol{i} = -\frac{\rho}{\epsilon_0}$. We have then the following fundamental relation :

$$\Delta V = \boldsymbol{i}$$

We are going to consider for some node $b \in N$ the following electrical flow denoted by $\boldsymbol{i}^b$:

$$\boldsymbol{i}(u) = \begin{cases} d(u) \text{ if } u \neq b \\ -\sum_{v \neq b} d(v) = -(vol(G) - d(b)) \text{ if } u = b \end{cases}$$

We denote by V^b the electrical potentials resulting from $\boldsymbol{i}^b$. The difference of potential $V_v^b - V_u^b$ will be denoted by V_{uv}^b.

We show in the next parts the following results:

1. (Appendix B.1) The electrical potentials are linked to the hypergraph Laplacian through the following relation for any a and b:
$$V_{ab}^a + V_{ba}^b = vol(\mathcal{H})(\Delta^\dagger_{a,a} + \Delta^\dagger_{b,b} - 2 \cdot \Delta^\dagger_{a,b}) = vol(\mathcal{H})\Omega_{a,b}$$
2. (Appendix B.2) The electrical potentials satisfy the following linear system for any a and b:
$$V_{ba}^b = \sum_{h \ni a} \frac{w_h(a)}{d(a)} \left[1 + \sum_{n \in h, n \neq a} \epsilon_h(n)\epsilon_h(a) \sqrt{\frac{w_h(n)}{w_h(a)}} V_{bn}^b \right].$$
with $V_{bb}^b = 0$

B.1 Link between $V_{ab}^a + V_{ba}^b$ and the Laplacian kernel $\Delta^\dagger$

We are going to consider for some node $b \in N$ the following electrical flow in the network ([6]):

$$\boldsymbol{i}(u) = \begin{cases} d(u) \text{ if } u \neq b \\ -\sum_{v \neq b} d(v) = -(vol(G) - d(b)) \text{ if } u = b \end{cases}$$

The corresponding electric potentials are denoted by V^b. Since the hypergraph is connected we have $Ker(\Delta) = Span(\mathbf{1})$ so $\mathbb{R}^{|N|} = Span(\mathbf{1}) \overset{\perp}{\oplus} Ker(\Delta)^\perp$. We look for the solution of $\Delta V = \boldsymbol{i}$ with $V = \mu\mathbf{1} + x$ where $x \in Ker(\Delta)^\perp$. Let's consider the operator $Q = \Delta^\dagger\Delta$. Q is the orthogonal projector operator on $Ker(\Delta)^\perp$ (general property of the pseudo-inverse). Since $x \in Ker(\Delta)^\perp$, we have :

$$\begin{aligned} x &= Qx \\ &= \Delta^\dagger \Delta x \\ &= \Delta^\dagger \boldsymbol{i} \end{aligned}$$

Since x satisfies $\Delta x = I$. Finally, we have $V = \Delta^\dagger I + \mu\mathbf{1}$. V is defined up to an additive constant. We are going to consider difference of potentials in the following. We have then:

$$\begin{aligned} V_{ba}^b = V_a^b - V_b^b &= (e_a - e_b)^T \Delta^\dagger \boldsymbol{i} \\ &= \sum_{u \neq b} d(u) \cdot (\Delta^\dagger_{u,a} - \Delta^\dagger_{u,b}) - (vol(G) - d(b)) \cdot (\Delta^\dagger_{a,b} - \Delta^\dagger_{b,b}) \\ &= \sum_{u \neq b,a} d(u) \cdot (\Delta^\dagger_{u,a} - \Delta^\dagger_{u,b}) + d(a) \cdot (\Delta^\dagger_{a,a} - \Delta^\dagger_{a,b}) + (vol(G) - d(b)) \cdot (\Delta^\dagger_{b,b} - \Delta^\dagger_{a,b}) \end{aligned}$$

By inverting a and b we get conversely:

$$V_{ab}^a = \sum_{u \neq b,a} d(u) \cdot (\Delta_{u,b}^\dagger - \Delta_{u,a}^\dagger) + d(b) \cdot (\Delta_{b,b}^\dagger - \Delta_{b,a}^\dagger) + (vol(G) - d(a)) \cdot (\Delta_{a,a}^\dagger - \Delta_{a,b}^\dagger)$$

and finally:

$$\begin{aligned} V_{ab}^a + V_{ba}^b &= vol(G) \cdot (\Delta_{a,a}^\dagger + \Delta_{b,b}^\dagger - 2 \cdot \Delta_{a,b}^\dagger) \\ &\quad + \sum_{u \neq b,a} d(u) \cdot \underbrace{(\Delta_{u,a}^\dagger - \Delta_{u,b}^\dagger + \Delta_{u,b}^\dagger - \Delta_{u,a}^\dagger)}_{=0} \\ &\quad + (\Delta_{a,a}^\dagger - \Delta_{a,b}^\dagger) \cdot \underbrace{(d(a) - d(a))}_{=0} + (\Delta_{b,b}^\dagger - \Delta_{b,a}^\dagger) \cdot \underbrace{(d(b) - d(b))}_{=0} \\ &= vol(G)(\Delta_{a,a}^\dagger + \Delta_{b,b}^\dagger - 2 \cdot \Delta_{a,b}^\dagger) \end{aligned}$$

This proof is valid for any system of nodes (graph or hypergraph) embedded with a Laplacian matrix Δ (related to some gradient $grad$).

B.2 Specific case of an hypergraph embedded with an unnormalized Laplacian

We start with the fundamental relations written for the node a:

$$e_a^T \Delta V^b = d(a)$$

,

We have:

$$\begin{aligned} \langle e_a, \Delta V^b \rangle &= \langle grad(e_a), grad(V^b) \rangle \\ &= \langle grad(e_a), grad(V^b - V_b^b) \rangle \end{aligned}$$

where V_b^b is the constant function on the nodes giving the value of the electrical potential in the node b. The unnormalized gradient of a constant function is null because of the weight balance condition.

$$\begin{aligned} \langle grad(e_a), grad(V^b - V_b^b) \rangle &= \sum_{h \ni a} \sqrt{w_h(a)} \left\{ \sum_{n \in h} (-P_h(n,a)) \sqrt{w_h(n)} \cdot (V_n^b - V_b^b) \right\} \\ &= \sum_{h \ni a} \sqrt{w_h(a)} \left\{ \sum_{n \in h} (-P_h(n,a)) \sqrt{w_h(n)} V_{bn}^b \right\} \end{aligned}$$

We get finally:

$$
\begin{aligned}
d(a) &= \sum_{h \ni a} \sqrt{w_h(a)} \left\{ \sum_{n \in h} (-P_h(n,a)) \sqrt{w_h(n)} V^b_{bn} \right\} \\
&= V^b_{ba} \underbrace{\sum_{h \setminus\ a \in h} w_h(a)(-P_h(a,a))}_{=d(a)} + \sum_{h \ni a} \sqrt{w_h(a)} \left\{ \sum_{n \in h, n \neq a} (-P_h(n,a)) \sqrt{w_h(n)} V^b_{bn} \right\}
\end{aligned}
$$

We can rewrite the last equality as a linear system:

$$
\begin{aligned}
V^b_{ba} &= 1 + \sum_{h \ni a} \frac{\sqrt{w_h(a)}}{d(a)} \left\{ \sum_{n \in h, n \neq a} P_h(n,a) \sqrt{w_h(n)} V^b_{bn} \right\} \\
&= \sum_{h \ni a} \frac{w_h(a)}{d(a)} \left\{ 1 + \sum_{n \in h, n \neq a} P_h(n,a) \sqrt{\frac{w_h(n)}{w_h(a)}} V^b_{b,n} \right\}
\end{aligned}
$$

which concludes the proof.

B.3 Graph harmonics and behavior of laplacian distances

As stated in part 2, the distance defined by an unnormalized laplacian Δ might not be a proper distance. Indeed, this distance is symmetric and always satisfies the triangle inequality but can have $\Omega_{i,j} = 0$ for $i \neq j$. Our goal in this part is to give more information about this pathological case.

First, let's write $\Delta = V \Lambda V^T$ where Λ is the diagonal matrix of the eigenvalues $\lambda_1 = 0 \leq \cdots \leq \lambda_{|N|}$ and V is an orthogonal matrix (which columns are the eigenvectors of Δ). We have $\Omega_{i,j} = 0$ if and only if the distance is null in the *V-space* between $\phi(i)$ and $\phi(j)$:

$$
\|\phi(i) - \phi(j)\|^2 = \sum_k \lambda_k^\dagger |V_{i,k} - V_{j,k}|^2 = 0
$$

As a consequence, we have to satisfy the following condition:

$$
V_{i,k} = V_{j,k} \text{ as soon as } \lambda_k \neq 0 \quad (*)
$$

In the following, we will assume without loss of generality that we have p null eigenvalues $\lambda_1 = \cdots = \lambda_p = 0$.

When the hypergraph is connected we have $Ker(\Delta) = Span(\mathbf{1})$ so $V_{i,1} = V_{j,1}$. Then, if $(*)$ is satisfied then the i-th and j-th lines of V are fully equals, which is impossible since V is orthogonal. In this case the resistance distance is a *proper distance*. It is the case for the classic connected graphs.

When the rank of Δ is lower, $(*)$ can be satisfied, which lead to the pathological case $\Omega_{i,j} = 0$ with $i \neq j$. We know that x is in $Ker(\Delta)$ if and only if, it defines an harmonic function for the gradient G:

$$\Delta x = 0 \Leftrightarrow x^T \Delta x = 0 \Leftrightarrow x^T G^T G x = 0 \Leftrightarrow \|Gx\|^2 = 0 \Leftrightarrow x \in Ker(G)$$

(the first equivalence is due to the fact that if $x^T \Delta x = 0$ then x is a minimum of the function $t \to t^T \Delta t$ and the first order condition gives $\Delta t = 0$).

Ideally, we should try to keep $Ker(\Delta)$ as small as possible. In some cases, we can put some hard constraint in order to have $rank(\Delta) = |N| - 1$ (for example in the optimization problem proposed in section 4 ; this rank constraint can be handled quite efficiently using manifold optimization). For the combination methods described and experimented in this paper, having $rank = |N| - 1$ is not guaranteed at the end. However, we are deliberately using post-processing methods that does not create uselessly artificial harmonics (for example, we flip the negative eigenvalues instead of replacing them by 0).

C Remarks concerning the *ZHS* algorithm

The purpose on this annex is to prove the validity of the algorithm *ZHS* when it uses an unnormalized Laplacian.

The goal of the algorithm is to minimize $J(f) = f^T \Delta f + \mu\|f - y\|$. The regularization term $f^T \Delta f = \sum_{[u,v]\in E} [grad(f)([u,v])]^2$ controls the smoothness of f on the graph. In [23], the authors use the normalized gradient and therefore the normalized Laplacian (see section 5. of the paper). However, the algorithm remains valid with other choices.

If we come back to the necessary condition used by [23]:

$$\begin{aligned} &\Delta f + \mu(f - y) = 0 \\ \Leftrightarrow \quad &(\Delta + \mu I)f \quad = \mu y \end{aligned}$$

$\Delta + \mu I$ is invertible for any $\mu > 0$ since $det(\Delta - \lambda I) \neq 0$ for any negative λ (Δ is semi-definite positive). If we write $\Delta = I - \theta$, we have:

$$\Delta + \mu I = (1 + \mu) \cdot I - \theta = (1 + \mu) \cdot \left(I - \frac{1}{1 + \mu} \cdot \theta \right) = (1 + \mu) \left(I - \alpha\theta \right)$$

,

where $\alpha = \frac{1}{1+\mu}$. The matrix $I - \alpha\theta$ is invertible for any $\mu > 0$ ($\Delta + \mu I$ is invertible and $1 + \mu \neq 0$ in this case). Since $\frac{\mu}{1+\mu} = 1 - \alpha$, we finally get:

$$f = (1 - \alpha)(I - \alpha\theta)^{-1} y$$

This formula is analog to the one proposed in [23] (theorem 3.3) and is valid for any Laplacian Δ. We are going to apply this algorithm with our unnormalized Laplacians.